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WAVES IN A ROD UNDER PULSED LOADING

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Wave processes in a semi-infinite rod located in an elastic medium under pulsed loading by an external distributed force are considered. A system of two differential equations of motion of Timoshenko's beam theory is solved with the use of the Laplace transform in time. The resultant integrals are determined numerically. The changes in bending and bending moment over the longitudinal coordinate at different times are demonstrated.

Key words: *stress and strain waves, rod, Laplace transform.*

Wave solutions of problems have found some recent applications; hence, an important problem is to determine the stress and strain waves in extended engineering structures under dynamic loading. For instance, a rod can be considered as a model of an underground long-distance pipeline. Knowing the time interval between the arrivals of the longitudinal and transverse waves, one can determine the distance between the center of the disturbance and the detecting station.

Problems similar to those considered below were solved in [1, 2], where the force and kinematic factors were assumed to be applied to the end face of an isolated rod.

The origin of the coordinate system is placed to the initial section of a semi-infinite rod, and the x axis is directed along the rod centerline. The external load is described by the expression

$$p(x,t) = \begin{cases} p_0(1 - x/a)H(t), & 0 \leqslant x \leqslant a, \quad t > 0, \\ 0, & x > a, \quad t < 0, \end{cases}$$

where t is the time and H is the Heaviside function:

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases}$$

The soil surrounding the pipeline is modeled by Winkler's foundation; the soil resistance is assumed to be proportional to bending:

$$p_* = \alpha W$$

(α is the foundation rigidity and W is the bending). The coefficient α for soils is found by the formula [3]

$$\alpha = 0.12E_*(b/l_0)^{1/2}/(1 - \mu_*^2).$$

Here E_* and μ_* are Young's modulus and Poisson's ratio for the soil, respectively, b is the width of the rod cross section, and l_0 is the unit length.

Using the dimensionless quantities

$$\xi = \frac{x}{r}, \quad w = \frac{W}{r}, \quad \tau = \frac{c_1 t}{r}, \quad m = \frac{Mr}{EJ}, \quad \beta = \frac{r}{a}, \quad r^2 = \frac{J}{F}$$

(r is the radius of inertia and F and J are the area and moment of inertia of the rod cross section) and taking into account the shear strain and rotation inertia, we write the equations of motions in displacements in dimensional form [4]:

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$$\frac{\partial^2 w}{\partial \xi^2} - \frac{\partial \theta}{\partial \xi} - \gamma \frac{\partial^2 w}{\partial \tau^2} - \zeta w = -k(1-\beta\xi)H(\tau),$$
$$\frac{\partial w}{\partial \xi} - \theta + \gamma\Big(\frac{\partial^2 \theta}{\partial \xi^2} - \frac{\partial^2 \theta}{\partial \tau^2}\Big) = 0. \tag{1}$$

Here $\gamma = c_1^2/c_2^2$, $\zeta = r^2\alpha/(\rho F c_2^2)$, $k = p_0 r/(\rho F c_2^2)$, $c_1^2 = E/\rho$, $c_2^2 = k'G/\rho$, c_1 and c_2 are the velocities of propagation of the longitudinal and transverse waves, θ is the angle of rotation of the section due to the bending moment, ρ, E, and G are the density, modulus of elasticity, and shear modulus of the rod material, and k' is a coefficient depending on the cross-sectional shape of the rod ($k' = 1.2$ for a rectangular cross section and $k' = 1.1$ for a circular cross section).

Applying the Laplace transform in time to system (1), we obtain

$$\frac{d^2 \bar{w}}{d\xi^2} - \frac{d\bar{\theta}}{d\xi} - (\gamma s^2 + \zeta)\bar{w} = -(1-\beta\xi)D,$$
$$\frac{d\bar{w}}{d\xi} + \gamma \frac{d^2\bar{\theta}}{d\xi^2} - (\gamma s^2 + 1)\bar{\theta} = 0. \tag{2}$$

Here $D = k/s$, s is the transform parameter, and $\bar{w}$ and $\bar{\theta}$ are the images of the functions w and θ. Eliminating $\bar{\theta}$ from system (2), we find

$$\frac{d^4\bar{w}}{d\xi^4} - (\gamma s^2 + s^2 + \zeta)\frac{d^2\bar{w}}{d\xi^2} + (\gamma s^2 + 1)\Big(s^2 + \frac{\zeta}{\gamma}\Big)\bar{w} = (1-\beta\xi)D\Big(s^2 + \frac{1}{\gamma}\Big),$$
$$\bar{\theta} = \frac{\gamma}{\gamma s^2 + 1}\Big[\frac{d^3\bar{w}}{d\xi^3} - \Big(\gamma s^2 + \zeta - \frac{1}{\gamma}\Big)\frac{d\bar{w}}{d\xi} - \beta D\Big]. \tag{3}$$

The solution of system (3) has the form

$$\bar{w} = A_1 \,\mathrm{e}^{-\lambda_1\xi} + A_2\,\mathrm{e}^{-\lambda_2\xi} + \frac{(1-\beta\xi)D}{\gamma s^2 + \zeta},$$
$$\bar{\theta} = (\lambda_1^2 - \gamma s^2 - \zeta)\frac{A_1}{-\lambda_1}\,\mathrm{e}^{-\lambda_1\xi} + (\lambda_2^2 - \gamma s^2 - \zeta)\frac{A_2}{-\lambda_2}\,\mathrm{e}^{-\lambda_2\xi} - \frac{\beta D}{(\gamma s^2+\zeta)(\gamma s^2 + 1)}, \tag{4}$$

where A_1 and A_2 are constants of integration of the first equation in system (3) and $\lambda_{1,2}$ are two (out of four) roots of the characteristic equation

$$\lambda^4 - [(\gamma+1)s^2 + \zeta]\lambda^2 + (\gamma s^2 + \zeta)(s^2 + 1/\gamma) = 0 \tag{5}$$

satisfying the condition of decay of $\bar{w}$ and $\bar{\theta}$ at infinity. The roots $\lambda_{1,2}$ of Eq. (5) are presented as

$$\lambda_{1,2}^2 = \frac{(\gamma+1)s^2 + \zeta}{2} \pm \frac{f}{a},$$

where $f = \sqrt{(s^2 - a_1^2)(s^2 - a_2^2)}$, $a_{1,2}^2 = (a/2)[a - \zeta \pm a\sqrt{1 - (\gamma-1)\zeta/\gamma}\,]$, and $a = 2/(\gamma - 1)$.

We consider two types of support in the initial section: hinge-supported and clamped rods. For a hinge-supported rod, the boundary conditions for the images have the form

$$\xi = 0: \qquad \bar{w}(0,\tau) = \frac{\partial\bar{\theta}(0,\tau)}{\partial\xi} = 0,$$

and the integration constants are

$$A_1 = -D\,\frac{\lambda_2^2 - \gamma s^2 - \zeta}{(\lambda_2^2 - \lambda_1^2)(\gamma s^2 + \zeta)}, \qquad A_2 = D\,\frac{\lambda_1^2 - \gamma s^2 - \zeta}{(\lambda_2^2 - \lambda_1^2)(\gamma s^2+\zeta)}. \tag{6}$$

For a clamped rod, the angles of rotation and shear are equal to zero

$$\xi = 0: \qquad \bar{\theta}(0,\tau) = 0, \quad \frac{\partial\bar{w}(0,\tau)}{\partial\xi} - \bar{\theta}(0,\tau) = 0,$$

and the integration constants are

$$A_1 = \frac{\beta\lambda_1 D}{(\lambda_2^2 - \lambda_1^2)(\gamma s^2 + 1)} \Big[1 + \frac{\lambda_2^2}{\gamma s^2 + \zeta}\Big(\frac{1}{\gamma s^2 + \zeta} - 1\Big)\Big],$$

$$A_2 = -\frac{\beta\lambda_2 D}{(\lambda_2^2 - \lambda_1^2)(\gamma s^2 + 1)} \Big[1 + \frac{\lambda_1^2}{\gamma s^2 + \zeta}\Big(\frac{1}{\gamma s^2 + \zeta} - 1\Big)\Big].$$

The bending moment is determined by the formula

$$m = \frac{d\theta}{d\xi}. \tag{7}$$

For a hinge-supported rod, Eqs. (4), (6), and (7) yield

$$\bar{w} = \frac{k}{s(\gamma s^2 + \zeta)} \Big(1 - \beta\xi - \frac{1}{2}\,(\mathrm{e}^{-\lambda_1\xi} + \mathrm{e}^{-\lambda_2\xi}) - \frac{2s^2 + a}{4f}\,(\mathrm{e}^{-\lambda_1\xi} - \mathrm{e}^{-\lambda_2\xi})\Big),$$
$$\bar{m} = \frac{ak}{2\gamma f s}\,(\mathrm{e}^{-\lambda_1\xi} - \mathrm{e}^{-\lambda_2\xi}). \tag{8}$$

The original functions are determined by the inversion formula

$$\frac{1}{2\pi i}\int_L F(s)\,\mathrm{e}^{\tau s}\,ds = \begin{cases} f(\tau), & \tau > 0, \\ 0, & \tau < 0, \end{cases}$$

where L is the Bromwich contour, which is a vertical straight line with the abscissa $c > 0$ and the ordinate tending to infinity.

To pass to real integrals, we present expressions (8) as

$$w(\xi,\tau) = \frac{k}{2\pi i}\,(I_0 + I_1 + I_2), \qquad m(\xi,\tau) = \frac{ak}{4\pi i\gamma}\,(I_4 - I_3). \tag{9}$$

Here

$$I_1 = I_1' + I_1'', \qquad I_2 = I_2' + I_2'', \qquad I_0 = \int_L \frac{(1 - \beta\xi)\,\mathrm{e}^{\tau s}}{s(\gamma s^2 + \zeta)}\,ds,$$

$$I_1' = -\int_L \frac{\mathrm{e}^{\tau s - \lambda_2\xi}}{2s(\gamma s^2 + \zeta)}\,ds, \quad I_1'' = \int_L \frac{(2s^2 + a)\,\mathrm{e}^{\tau s - \lambda_2\xi}}{4fs(\gamma s^2 + \zeta)}\,ds, \quad I_2' = -\int_L \frac{\mathrm{e}^{\tau s - \lambda_1\xi}}{2s(\gamma s^2 + \zeta)}\,ds,$$

$$I_2'' = -\int_L \frac{(2s^2 + a)\,\mathrm{e}^{\tau s - \lambda_1\xi}}{4fs(\gamma s^2 + \zeta)}\,ds, \quad I_3 = -\int_L \frac{\mathrm{e}^{\tau s - \lambda_2\xi}}{fs}\,ds, \quad I_4 = \int_L \frac{\mathrm{e}^{\tau s - \lambda_1\xi}}{fs}\,ds.$$

As $s \to \infty$, all integrands in (9) tend to zero, while λ_1 and λ_2 tend to constant values:

$$\lim_{s\to\infty}(\lambda_2/s) = 1, \qquad \lim_{s\to\infty}(\lambda_1/s) = \sqrt{\gamma}\,. \tag{10}$$

The inversion formula and Eq. (10) imply that disturbances propagate along the rod in the form of two waves with velocities c_1 and c_2. The domain of disturbance propagation is divided by wave fronts into two parts. At an arbitrary time τ, the coordinates of the fronts are determined as $\xi_1 = \tau$ and $\xi_2 = \tau/\sqrt{\gamma}$. The domain $0 < \xi \leqslant \xi_1$ is covered by the bending wave, and the wave parameters are determined by the integrals I_1 and I_3, which are not equal to zero if $\xi_1 < \tau$ and are equal to zero if $\xi_1 > \tau$. In the interval $0 \leqslant \xi \leqslant \xi_2$, there are both the bending and the shear waves; the parameters of the shear wave are determined by the integrals I_2 and I_4, which are not equal to zero if $\xi < \tau/\sqrt{\gamma}$ and are equal to zero if $\xi > \tau/\sqrt{\gamma}$. In the interval $0 \leqslant \xi \leqslant \xi_3 = 1/\beta$, there also arise strains caused by the action of external loading, which are determined by the integral I_0. The value of ξ_3 is constant and independent of time. Figure 1 shows the distribution of disturbances along the rod at a fixed time.

The simple poles and the points of branching of the integrands in (9) are listed in Table 1. The integration contours are shown in Fig. 2. The integration contour I_1' is shown by the dashed curve in Fig. 2a; this contour is the cut along an imaginary axis from the point $s = \pm ia_3$ to the point $s = \pm ia_4$ and further to the large-radius semi-circle.

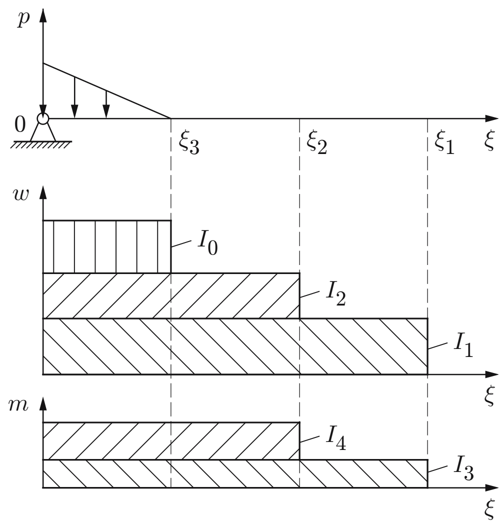

Fig. 1. Distribution of disturbances along the rod at a fixed time.

TABLE 1

Poles and Points of Branching of the Integrands in Eqs. (9)

Integral	Poles s	Point of branching s
I_0	$0;\ \pm ia_3$	—
I_1'	$0;\ \pm ia_3$	$\pm ia_3;\ \pm ia_4$
I_1''	$0;\ \pm ia_3$	$\pm a_1;\ \pm ia_2;\ \pm ia_3;\ \pm ia_4$
I_2'	$0;\ \pm ia_3$	—
I_2''	$0;\ \pm ia_3$	$\pm a_1;\ \pm ia_2$
I_3	0	$\pm a_1;\ \pm ia_2;\ \pm ia_3;\ \pm ia_4$
I_4	0	$\pm a_1;\ \pm ia_2$

Note. $a_3 = (\zeta/\gamma)^{1/2}$ and $a_4 = (1/\gamma)^{1/2}$.

The complex expressions in the integrands of Eqs. (9) were obtained in [5] with allowance for constraints on their arguments depending on the integration path. These expressions are listed in Table 2. The integrals in Eqs. (9) were calculated by the formula

$$I = \sum \operatorname{res}(s) - \sum \int\limits_{\gamma_i},$$

where γ_i are the integration paths in the positive direction along the cut edges and arcs of circles with an infinitesimal radius. As the radius of a small circle tends to zero, the length of the integration path and, correspondingly, the integral also tend to zero. The integrals over the segments of the integration contour denoted in Fig. 2 by 1 and I and over all cut edges along the imaginary axis are mutually eliminated. After calculating the integrals, we find

$$I_0 = I_1' = 0, \qquad I_1'' = -I_2'', \qquad I_3 = I_4.$$

The expressions for the bending and bending moment have the form

$$\begin{aligned} &w(\xi,\tau) = \operatorname{res}(s)_1 - I_1'', \qquad m(\xi,\tau) = -\operatorname{res}(s)_3 + I_3, \qquad \xi_2 \leqslant \xi \leqslant \xi_1, \\ &w(\xi,\tau) = \operatorname{res}(s)_1 + \operatorname{res}(s)_2, \qquad m(\xi,\tau) = -\operatorname{res}(s)_3 + \operatorname{res}(s)_4, \qquad 0 \leqslant \xi \leqslant \xi_2, \end{aligned} \tag{11}$$

where

$$\operatorname{res}(s)_1 = -k\Big[\frac{\mathrm{e}^{-\alpha_1\xi}}{2\zeta}\Big(\cos\beta_1\xi - \frac{1}{\nu\zeta}\,\sin\beta_1\xi\Big) + \frac{\gamma(\zeta-1)}{2\zeta^2}\,\cos a_3\xi\Big],$$

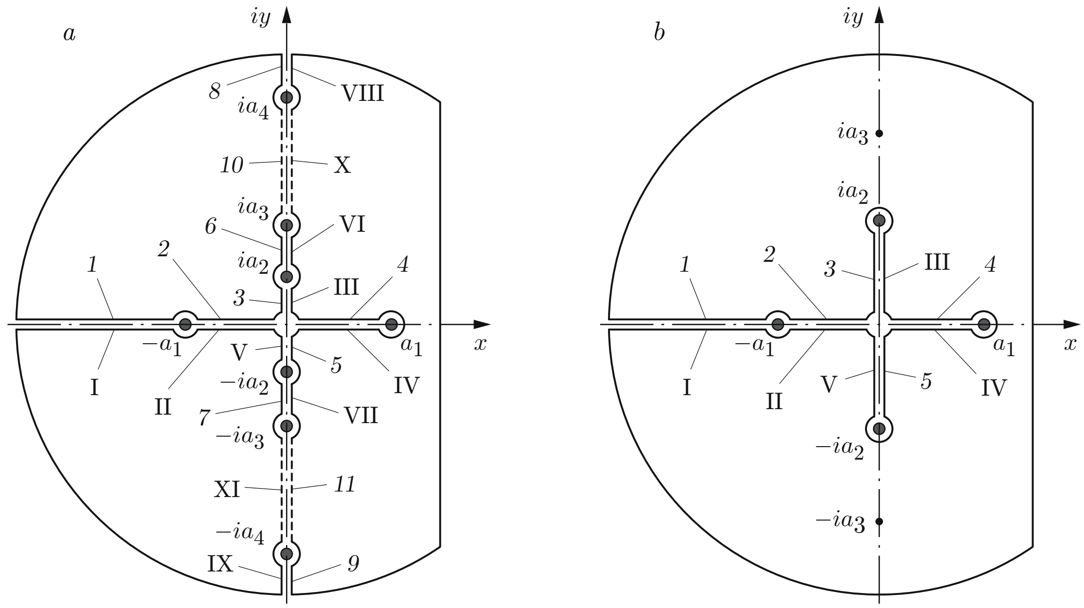

Fig. 2. Contours of integration of I_1, I_2 (a) and I_2'', I_4 (b): the opposite sides of the integration path are indicated by 1–11 and I–XI.

TABLE 2

Complex Quantities in the Integrands of the Integrals I_i $(i = 1, 2, 3, 4)$ for Different Integration Paths

Integration path	s	s^2	f	$\lambda_{1,2}$	Interval of variation of x and y
$\frac{1}{\mathrm{I}}$	$-x$	x^2	$-f_1^*$	$\sqrt{R_1 \mp R_2}$	$-\infty \leqslant x \leqslant -a_1$
$\frac{2}{\mathrm{II}}$	$-x$	x^2	$\pm i f_2^*$	$\frac{\eta_1 \pm i\eta_2}{\eta_1 \mp i\eta_2}$	$-a_1 \leqslant x \leqslant 0$
$\frac{4}{\mathrm{IV}}$	x	x^2	$\pm i f_2^*$	$\frac{\eta_2 \pm i\eta_1}{\eta_2 \mp i\eta_1}$	$0 \leqslant x \leqslant a_1$
$\frac{3}{\mathrm{III}}$	iy	$-y^2$	$i f_3^*$	$\eta_3 \pm i\eta_4$	$0 \leqslant y \leqslant a_2$
$\frac{5}{\mathrm{V}}$	$-iy$	$-y^2$	$-i f_3^*$	$\eta_3 \mp i\eta_4$	$-a_2 \leqslant y \leqslant 0$
$\frac{6,\ \mathrm{VI}}{7,\ \mathrm{VII}}$	$\pm iy$	$-y^2$	$-f_4^*$	$\lambda_1 = i\eta_5$ $\lambda_2 = \eta_6$	$\frac{y \geqslant a_2}{y \leqslant -a_2}$

Notes. 1) $f_1^* = \sqrt{(x^2 - a_1^2)(x^2 + a_2^2)}$, $f_2^* = \sqrt{(a_1^2 - x^2)(a_2^2 + x^2)}$, $f_3^* = \sqrt{(a_1^2 + y^2)(a_2^2 - y^2)}$, and $f_4^* = \sqrt{(y^2 + a_2^2)(y^2 - a_2^2)}$.

2) The Arabic and Roman numerals refer to the ends of the integration contours in Fig. 2.

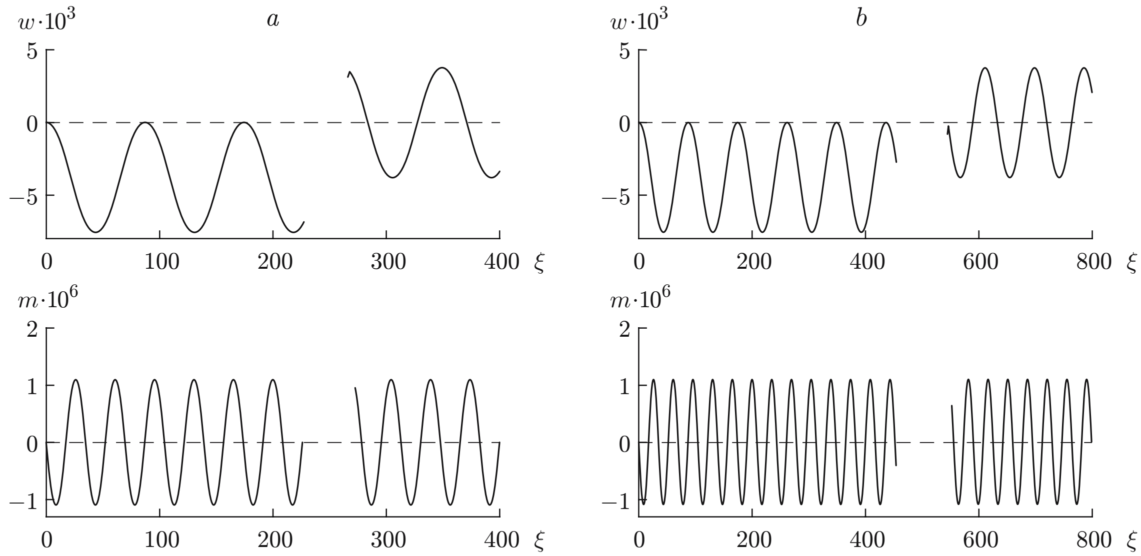

Fig. 3. Distributions of the bending and bending moment along the rod for $\tau = 400$ (a) and 800 (b).

$$\operatorname{res}(s)_2 = -k\Big[\frac{e^{-\alpha_1\xi}}{2\zeta}\Big(\cos\beta_1\xi + \frac{1}{\nu\zeta}\sin\beta_1\xi\Big) - \frac{1}{\zeta} - \frac{\gamma(\zeta-1)}{2\zeta^2}\Big],$$

$$\operatorname{res}(s)_3 = \frac{k}{\gamma\zeta\nu}\sin\beta_1\xi, \qquad \operatorname{res}(s)_4 = -\operatorname{res}(s)_3,$$

$$T_0 = e^{\tau x - \eta_2\xi}\cos\eta_1\xi - e^{-\tau x - \eta_1\xi}\cos\eta_2\xi,$$

$$I_1'' = -\frac{k}{4\pi}\int_0^{a_1}\frac{(2x^2+a)T_0}{xf_2^*(\gamma x^2+\zeta)}\,dx, \qquad I_3 = -\frac{ak}{2\pi\gamma}\int_0^{a_1}\frac{T_0}{xf_2^*}\,dx,$$

$$\alpha_1 = \sqrt{(a_3+\zeta/2)/2}, \qquad \beta_1 = \sqrt{(a_3-\zeta/2)/2}, \qquad \nu = \sqrt{4/(\gamma\zeta)-1},$$

and $\operatorname{res}(s)_i$ are the residues of the integrands in the integrals I_i ($i = 1,2,3,4$), respectively.

Let us consider a numerical example. We assume that the rod has a rectangular cross section, $b = h = 0.1$ m, $F = b \times h$, $p_0 = 10$ kN/m, $E = 2 \cdot 10^5$ MPa, $\rho = 8$ tons/m^3, $c_1 = 5 \cdot 10^3$ m/sec, $c_2 = 2.84 \cdot 10^3$ m/sec, $\zeta = 1.35 \cdot 10^{-2}$, $k = 0.45 \cdot 10^{-6}$, $\beta = 0.1$, $\gamma = 3.1$, $a = 0.95$, $a_1 = 0.94$, $a_2 = 0.051$, $a_3 = 0.072$, and $a_4 = 0.225$.

The calculations were performed by formulas (11). In the integral I_1'', the integrand has infinite discontinuities at the points $x = 0$ and $x = a_1$, and the function T_0 is oscillating; hence, more than 10 steps within the half-wave length were ensured during integration with respect to x. The lower limit was assumed to be δ, and the upper limit was $a_1(1-\delta)$, where $\delta = 10^{-15}$. Thus, the principal value of the improper integral was determined. The calculations were performed by the method of trapezoids.

Based on the calculated results, the dependences $w(\xi)$ and $m(\xi)$ for two times were plotted in Fig. 3.

It follows from the data presented that the changes in the bending and bending moment in the disturbed zone have an oscillating character. In the expression for w, the prevailing term is that containing the factor $\cos a_3\xi$. The periods of the functions $\cos a_3\xi$ and $\cos\beta_1\xi$ are $2\pi/a_3 = 87$ and $2\pi/\beta_1 = 35$ ($\beta = 0.18$), respectively. In this case, the quantities a_3 and β_1 may be considered as oscillation frequencies. The ratio of periodicities and frequencies of oscillations is $\beta_1/a_3 = 2.5$. The oscillations of the bending in the interval $0 \leqslant \xi \leqslant \xi_2$ occur around the strained state $w = -k\gamma(\zeta-1)/(2\zeta^2)$ with a frequency a_3; in the interval $\xi_2 \leqslant \xi \leqslant \xi_1$, the bending oscillates around the initial state with the same frequency. The bending moment oscillates around the initial state with a frequency β_1.

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